The factorization of a q-difference equation for continuous q-Hermite polynomials

2007 J. Phys. A: Math. Theor. 40 9311

(http://iopscience.iop.org/1751-8121/40/31/009)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 9311–9317 doi:10.1088/1751-8113/40/31/009

The factorization of a q-difference equation for continuous q-Hermite polynomials

M N Atakishiyev[1] **and A U Klimyk**[2]

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México, CP 62210 Cuernavaca, Morelos, Mexico
[2] Bogolyubov Institute for Theoretical Physics, Kiev 03143, Ukraine

E-mail: mamed@matcuer.unam.mx and aklimyk@bitp.kiev.ua

Received 11 January 2007, in final form 7 June 2007
Published 19 July 2007
Online at stacks.iop.org/JPhysA/40/9311

Abstract

We argue that a customary q-difference equation for the continuous q-Hermite polynomials $H_n(x|q)$ can be written in the factorized form as $\left[\left(\mathcal{D}_x^q\right)^2 - 1\right] H_n(x|q) = (q^{-n} - 1) H_n(x|q)$, where $\mathcal{D}_x^q$ is some explicitly known q-difference operator. This means that the polynomials $H_n(x|q)$ are in fact governed by the q-difference equation $\mathcal{D}_x^q H_n(x|q) = q^{-n/2} H_n(x|q)$, which is simpler than the conventional one. It is shown that a similar factorization holds for the continuous q^{-1}-Hermite polynomials $h_n(x|q)$.

PACS numbers: 02.30.Gp, 02.30.Tb, 03.65.Db

It is well known that the theory of q-Hermite polynomials is one of the main instruments for studying q-oscillators and their applications (see, for example, [1, 2]). In particular, q-difference equations for q-Hermite polynomials are known to be intimately connected with Hamiltonians of the corresponding systems. These equations are direct consequences of the eigenvalue problem for the corresponding invariant Casimir operators.

The continuous q-Hermite polynomials are also related to the oscillator representations (a special case of the discrete series representations) of the quantum algebra $\mathrm{su}_q(1, 1)$, which are constructed with the aid of the creation and annihilation operators for the q-oscillator (see, for example, [3]).

The aim of our paper is to study q-difference equations for the continuous q-Hermite polynomials for both cases when $0 < q < 1$ and $q > 1$. The continuous q-Hermite polynomials for $q > 1$ are essentially different from those for $0 < q < 1$. We derive factorizations for q-difference equations for both of these sets of polynomials. In fact, factorization for the continuous q-Hermite polynomials with $0 < q < 1$ (see equation (8) below) and with $q > 1$ (see (20)) means that the Casimir operator for q-oscillator algebra and the Casimir operator for representations of the quantum algebra $\mathrm{su}_q(1, 1)$, associated with q-Hermite polynomials, admit *dimidiation* (that is, the determination of square root). There is no such phenomenon in the cases of the standard quantum harmonic oscillator algebra and representations of the classical Lie algebra su(1, 1). It is well known that invariant Casimir operators in the q-case can be written in different (non-equivalent) forms. However, it was not known that Casimir operator admits evaluating its square root. Although a physical nature of this phenomenon is not yet clear, it seems worth while to call attention to this problem here.

The continuous q-Hermite polynomials of Rogers, $H_n(x|q)$, $0 < q < 1$, are orthogonal on the finite interval $-1 \leqslant x := \cos\theta \leqslant 1$,

$$\frac{1}{2\pi}\int_{-1}^{1} H_m(x|q)H_n(x|q)\widetilde{w}(x|q)\,\mathrm{d}x = \frac{\delta_{mn}}{(q^{n+1};q)_\infty}, \tag{1}$$

with respect to the weight function (we employ standard notations of the theory of special functions, see, for example, [4] or [5])

$$\widetilde{w}(x|q) := \frac{1}{\sin\theta}(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty. \tag{2}$$

These polynomials satisfy the q-difference equation

$$D_q[\widetilde{w}(x|q)D_q H_n(x|q)] = \frac{4q(1-q^{-n})}{(1-q)^2}H_n(x|q)\widetilde{w}(x|q), \tag{3}$$

written in a self-adjoint form [6]. The D_q in (3) is the conventional notation for the Askey–Wilson divided-difference operator defined as

$$\begin{aligned} &D_q f(x) := \frac{\delta_q f(x)}{\delta_q x}, \\ &\delta_q g(\mathrm{e}^{\mathrm{i}\theta}) := g(q^{1/2}\,\mathrm{e}^{\mathrm{i}\theta}) - g(q^{-1/2}\,\mathrm{e}^{\mathrm{i}\theta}), \qquad f(x) \equiv g(\mathrm{e}^{\mathrm{i}\theta}), \qquad x = \cos\theta. \end{aligned} \tag{4}$$

In what follows we find it more convenient to employ the explicit expression

$$D_q f(x) = \frac{\sqrt{q}}{\mathrm{i}(1-q)}\frac{1}{\sin\theta}\big(\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} - \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}\big)f(x), \qquad \partial_\theta \equiv \frac{\mathrm{d}}{\mathrm{d}\theta}, \tag{5}$$

for the D_q in terms of the shift operators (or the operators of the finite displacement, [7]) $\mathrm{e}^{\pm a\partial_\theta}g(\theta) := g(\theta \pm a)$ with respect to the variable θ. Although it is customary to represent q-difference equation for the q-Hermite polynomials in the self-adjoint form (3) (see [8, p 115]), one may eliminate the weight function $\widetilde{w}(x|q)$ from (3) by utilizing its property that

$$\exp\big(\pm\,\mathrm{i}\ln q^{1/2}\partial_\theta\big)\widetilde{w}(x|q) = -\frac{\mathrm{e}^{\pm 2\mathrm{i}\theta}}{\sqrt{q}}\widetilde{w}(x|q). \tag{6}$$

The validity of (6) is straightforward to verify upon using the explicit expression (2) for the weight function $\widetilde{w}(x|q)$.

Thus, combining (3) and (6) results in the q-difference equation

$$\frac{1}{2\mathrm{i}\sin\theta}\left[\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-q\,\mathrm{e}^{-2\mathrm{i}\theta}}(\mathrm{e}^{\mathrm{i}\ln q\partial_\theta}-1) + \frac{\mathrm{e}^{-\mathrm{i}\theta}}{1-q\,\mathrm{e}^{2\mathrm{i}\theta}}(1-\mathrm{e}^{-\mathrm{i}\ln q\partial_\theta})\right]H_n(x|q) = (q^{-n}-1)H_n(x|q) \tag{7}$$

for the continuous q-Hermite polynomials $H_n(x|q)$, which does not explicitly contain the weight function $\widetilde{w}(x|q)$.

In connection with equation (7) it should be remarked that Koornwinder [9] has recently studied in detail raising and lowering relations for the Askey–Wilson polynomials $p_n(x; a, b, c, d|q)$. We recall that the Askey–Wilson family for $a = b = c = d = 0$ is known to reduce to the continuous q-Hermite polynomials $H_n(x|q)$. So, as a consistency check, one may verify that (7) is in complete agreement with the particular case of the equation $Dp_n = \lambda_n p_n$ (i.e., equation (4.5) in [9]) for Askey–Wilson polynomials with vanishing parameters a, b, c, d.

We are now in a position to show that equation (7) admits a factorization. Indeed, with the help of two simple trigonometric identities

$$\frac{e^{\pm i\theta}}{2i\sin\theta} = \pm\frac{1}{1-e^{\mp 2i\theta}}$$

one can represent the left side of (7) as

$$\begin{aligned}
&\frac{1}{2i\sin\theta}\left(\frac{e^{i\theta}}{1-q\,e^{-2i\theta}}\,e^{i\ln q\,\partial_\theta} - \frac{e^{-i\theta}}{1-q\,e^{2i\theta}}\,e^{-i\ln q\,\partial_\theta} - \frac{e^{i\theta}}{1-q\,e^{-2i\theta}} + \frac{e^{-i\theta}}{1-q\,e^{2i\theta}}\right)H_n(x|q)\\
&\quad = \left[\frac{1}{1-e^{-2i\theta}}\,e^{i\ln q^{1/2}\partial_\theta}\,\frac{1}{1-e^{-2i\theta}}\,e^{i\ln q^{1/2}\partial_\theta} + \frac{1}{1-e^{2i\theta}}\,e^{-i\ln q^{1/2}\partial_\theta}\,\frac{1}{1-e^{2i\theta}}\,e^{-i\ln q^{1/2}\partial_\theta}\right.\\
&\qquad \left. + \frac{q(1+q)}{(1+q)^2-4qx^2} - 1\right]H_n(x|q), \qquad x = \cos\theta.
\end{aligned}$$

The expression in square brackets factorizes into a product $\left(\mathcal{D}_x^q+1\right)\left(\mathcal{D}_x^q-1\right)$ and the whole equation (7) may be written as

$$\left(\mathcal{D}_x^q\right)^2 H_n(x|q) = q^{-n}H_n(x|q), \tag{8}$$

where the q-difference operator $\mathcal{D}_x^q$ is

$$\begin{aligned}
\mathcal{D}_x^q &:= \frac{1}{1-e^{-2i\theta}}\,e^{i\ln q^{1/2}\partial_\theta} + \frac{1}{1-e^{2i\theta}}\,e^{-i\ln q^{1/2}\partial_\theta}\\
&\equiv \frac{1}{2i\sin\theta}\left(e^{i\theta}\,e^{i\ln q^{1/2}\partial_\theta} - e^{-i\theta}\,e^{-i\ln q^{1/2}\partial_\theta}\right).
\end{aligned} \tag{9}$$

To facilitate ease of clarifying the distinction between $\mathcal{D}_x^q$ and the Askey–Wilson divided-difference operator D_q, defined by (4), one may also write (9) in the form

$$\begin{aligned}
\mathcal{D}_x^q f(x) &= \frac{1-q}{2\sqrt{q}}\,\frac{1}{\delta_q x}\left[e^{-i\theta}g(q^{1/2}\,e^{i\theta}) - e^{i\theta}g(q^{-1/2}\,e^{i\theta})\right]\\
&= \frac{e^{i\theta}g\left(q^{-1/2}e^{i\theta}\right) - e^{-i\theta}g\left(q^{1/2}e^{i\theta}\right)}{e^{i\theta}-e^{-i\theta}},
\end{aligned} \tag{10}$$

where $g(e^{i\theta}) \equiv f(x)$ and $x = \cos\theta$, as before.

For various applications it is important that the D_q and $\widetilde{w}^{-1}(x|q)D_q\widetilde{w}(x|q)$ are in fact lowering and raising operators, respectively, for the continuous q-Hermite polynomials $H_n(x|q)$ (see, for example, [8, formulae (3.26.7) and (3.26.9)]). This circumstance enables one to interpret a Hilbert space of functions on $[-1,1]$, which are square integrable with respect to the weight $\widetilde{w}(x|q)$, as a direct sum of two $su_q(1,1)$-irreducible subspaces $T^+_{1/4}$ and $T^+_{3/4}$, consisting of even and odd functions, respectively [3, 10]. So it becomes transparent how the Askey–Wilson divided-difference operator D_q and the continuous q-Hermite polynomials $H_n(x|q)$ are interrelated from the group-theoretic point of view.

An explicit analytic relation between the Askey–Wilson divided-difference operator D_q and the difference operator $\mathcal{D}_x^q$, which surfaces in (8), involves the so-called *averaging difference operator* $\mathcal{A}_q$, defined as

$$(\mathcal{A}_q f)(x) = \tfrac{1}{2}\left(e^{i\ln q^{1/2}\partial_\theta} + e^{-i\ln q^{1/2}\partial_\theta}\right)f(x) \equiv \cos\left(\ln q^{1/2}\partial_\theta\right)f(x). \tag{11}$$

We recall that the averaging operator $\mathcal{A}_q$ is intimately associated with the Askey–Wilson operator D_q because the product rule for the latter one is of the form (see, for example, formula (21.6.4) in [11])

$$D_q f(x)g(x) = \mathcal{A}_q f(x)D_q g(x) + D_q f(x)\mathcal{A}_q g(x).$$

So, from (4), (9) and (11) one concludes that the $\mathcal{D}_x^q$ is may be expressed in terms of the known operators D_q and $\mathcal{A}_q$ as

$$\mathcal{D}_x^q = \mathcal{A}_q + \frac{1-q}{2\sqrt{q}} x D_q. \tag{12}$$

Note that the operator $\left(\mathcal{D}_x^q\right)^2$ represents, as equation (8) implies, an unbounded operator on the Hilbert space $L^2(-1,1)$ with the scalar product

$$\langle g_1, g_2\rangle = \frac{1}{2\pi}\int_{-1}^{1} g_1(x)\overline{g_2(x)}\widetilde{w}(x|q)\,\mathrm{d}x, \tag{13}$$

where the weight function $\widetilde{w}(x|q)$ is defined by (2). In view of (1) the polynomials $p_n(x) := (q^{n+1};q)_\infty^{-1/2} H_n(x|q), n = 0,1,2,\ldots,$ constitute an orthonormal basis in this space such that $\left(\mathcal{D}_x^q\right)^2 p_n(x) = q^{-n} p_n(x)$. In particular, the operator $\left(\mathcal{D}_x^q\right)^2$ is defined on the linear span $\mathcal{H}$ of the basis functions $p_n(x)$, which is everywhere dense in $L^2(-1,1)$. We close $\left(\mathcal{D}_x^q\right)^2$ with respect to the scalar product (13). Since $\left(\mathcal{D}_x^q\right)^2$ is diagonal with respect to the orthonormal basis $p_n(x), n = 0,1,2,\ldots,$ its closure $\overline{\left(\mathcal{D}_x^q\right)^2}$ is a self-adjoint operator which coincides on $\mathcal{H}$ with $\left(\mathcal{D}_x^q\right)^2$. According to the theory of self-adjoint operators (see [12, chapter 6]), we can take a square root of the operator $\overline{\left(\mathcal{D}_x^q\right)^2}$. This square root is a self-adjoint operator too and has the same eigenfunctions as the operator $\overline{\left(\mathcal{D}_x^q\right)^2}$ does. We denote this operator by $\overline{\mathcal{D}_x^q}$. It is evident that on the subspace $\mathcal{H}$ the operator $\overline{\mathcal{D}_x^q}$ coincides with the $\mathcal{D}_x^q$. That is, the $\mathcal{D}_x^q$ is a well-defined operator on the Hilbert space $L^2(-1,1)$ with an everywhere dense subspace of definition. Moreover, according to the definition of a function of a self-adjoint operator (see [12, chapter 6]), we have $\overline{\mathcal{D}_x^q}\, p_n(x) = q^{-n/2} p_n(x)$, that is

$$\mathcal{D}_x^q H_n(x|q) \equiv \left[\mathcal{A}_q + \frac{1-q}{2\sqrt{q}} x D_q\right] H_n(x|q) = q^{-n/2} H_n(x|q). \tag{14}$$

Thus, the continuous q-Hermite polynomials are in fact governed by a simpler q-difference equation (14) which is, in essence, a factorized form of (8).

This is a place to point out that the first explicit statement of equation (14), that we know, is in [13] and [14]: in the paper [13], it was stated without proof, whereas in [14] it was proved by employing the Rogers generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(x|q) = (t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1} \tag{15}$$

for the continuous q-Hermite polynomials $H_n(x;q)$ (see [4, p 26]) as follows. Apply the q-difference operator $\mathcal{D}_x^q$ to both sides of the generating function (15) to derive that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \mathcal{D}_x^q H_n(x|q) &= \mathcal{D}_x^q (t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1} \\ &= (q^{-1/2} t\,\mathrm{e}^{\mathrm{i}\theta}, q^{-1/2} t\,\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} q^{-n/2} H_n(x|q).\end{aligned}$$

Then equate coefficients of the same powers of t on the extremal sides above, to obtain the proof that equation (14) is consistent with the generating function (15). However, it should be noted that neither [13] nor [14] does contain any discussion of connection between q-difference equations (14) and (3) or (7).

In the limit as $q \to 1$ the continuous q-Hermite polynomials $H_n(x|q)$ are known to reduce to the ordinary Hermite polynomials $H_n(x)$ (see, for example, [8, p 144]),

$$\lim_{q\to 1} \kappa^{-n} H_n(\kappa x|q) = H_n(x), \qquad \kappa := \sqrt{\frac{1-q}{2}}. \tag{16}$$

Hence, if one rescales $x \to \kappa x$ and then lets $q \to 1$ in q-difference equations (8) and (14), both of these equations reduce to the same second-order differential equation

$$\left(\partial_x^2 - 2x\partial_x + 2n\right) H_n(x) = 0, \qquad \partial_x \equiv \frac{\mathrm{d}}{\mathrm{d}x},$$

for the ordinary Hermite polynomials $H_n(x)$. This fact is an immediate consequence of the limit property

$$\lim_{q\to 1} \left[\frac{1}{1-q}\left(\mathcal{D}^q_{\kappa x} - I\right)\right] = \frac{1}{2}\left(x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\right)\frac{\mathrm{d}}{\mathrm{d}x} \tag{17}$$

of the q-difference operator $\mathcal{D}^q_x$, which is straightforward checked by employing its definition (9) or (10). Note that the rescaling parameter κ in (17) is the same as in (16), whereas I is the identity operator.

Observe also that by combining (14) and (6) one arrives at the q-difference equation

$$\mathcal{D}^{1/q}_x H_n(x|q)\widetilde{w}(x|q) = q^{-(n+1)/2} H_n(x|q)\widetilde{w}(x|q), \tag{18}$$

which can be viewed as a factorized form of the conventional q-difference equation (3).

In the foregoing exposition up to the present point it has been implied that $0 < q < 1$. Of course, the case of $q > 1$ can be treated in a similar way. We briefly state below some explicit formulae for the case of $q > 1$ too. As was noted by Askey [15], one should deal with the case of the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers when $q > 1$ by introducing a family of polynomials

$$h_n(x|q) := \mathrm{i}^{-n} H_n(\mathrm{i}x|q^{-1}), \tag{19}$$

which are called the continuous q^{-1}-Hermite polynomials [16]. So the transformation $q \to q^{-1}$ and the change of variables $\theta = \pi/2 - \mathrm{i}\varphi$ in the q-difference equation (14) converts it, on account of definition (19), into equation

$$\widetilde{\mathcal{D}}^q_x h_n(x|q) = q^{n/2} h_n(x|q), \qquad x = \sinh\varphi, \tag{20}$$

where the q-difference operator $\widetilde{\mathcal{D}}^q_x$ is of the form

$$\widetilde{\mathcal{D}}^q_x := \frac{1}{2\cosh\varphi}\left(\mathrm{e}^{\varphi}\,\mathrm{e}^{\ln q^{1/2}\partial_\varphi} + \mathrm{e}^{-\varphi}\,\mathrm{e}^{-\ln q^{1/2}\partial_\varphi}\right). \tag{21}$$

The advantage of a compact form of equation (20) becomes transparent upon comparing it with difference equation (21.6.9) for the continuous q^{-1}-Hermite polynomials $h_n(x|q)$ in [11]. It should also be noted that q-difference equation (20) is consistent with the generating function

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} t^n h_n(\sinh\varphi|q) = (t\,\mathrm{e}^{-\varphi}, -t\,\mathrm{e}^{\varphi}; q)_\infty$$

for the continuous q^{-1}-Hermite polynomials $h_n(x|q)$ [16].

The direct proof of (20) follows the same lines as the proof of (14) using the results of the appendix of the paper [17]. Namely, an analogue of the q-difference equation (3) is obtained from the raising and lowering operators (A.1) and (A.2) in [17]. Formula (A.10) in [17] gives an analogue of relation (6). An analogue of the Hilbert space $L^2(-1,1)$, associated with $0 < q < 1$, in this case is constructed in the following way. We take the Hilbert space $L^2(\mathbb{R})$ with the scalar product determined by formula (3.2) in [15]. The polynomials

$h_n(x), n = 0, 1, 2, \ldots$, are orthogonal in this Hilbert space. However, this set of polynomials does not constitute a basis of $L^2(\mathbb{R})$ since the orthogonality measure in (3.2) of [15] is not extremal for the continuous q^{-1}-Hermite polynomials. For this reason, we create the closed subspace $\mathcal{L}$ of $L^2(\mathbb{R})$ spanned by the polynomials $h_n(x), n = 0, 1, 2, \ldots$. It is shown in the same way as above that $\left(\widetilde{\mathcal{D}}_x^q\right)^2$ is a bounded self-adjoint operator on the Hilbert space $\mathcal{L}$, diagonalizable by the polynomials $h_n(x|q), n = 0, 1, 2, \ldots$. Therefore, relation (19) holds for the operator $\widetilde{\mathcal{D}}_x^q$.

In conclusion, this short paper should be considered as an attempt to call attention to a curious fact that the conventional q-difference equation (7) for the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers admits factorization of the form $\left[\left(\mathcal{D}_x^q\right)^2 - 1\right]H_n(x|q) = (q^{-n} - 1)H_n(x|q)$, where $\mathcal{D}_x^q$ is defined by (9). This circumstance seems to have escaped the note of all those with whom we share interests in q-special functions.

Since the continuous q-Hermite polynomials $H_n(x|q)$ occupy the lowest level in the hierarchy of ${}_4\phi_3$ polynomials with positive orthogonality measures, it is of interest to find out whether there are instances from higher levels in the Askey q-scheme [8], which also admit factorization of an appropriate q-difference equation.

It is well known that the q-difference equation (3) is related to the Hamiltonian for the q-oscillator [10]. So it would be of interest to look for some insight into equations (14), (18) and (20) physically.

As mentioned at the beginning of the paper, factorizations of q-difference equations for the continuous q-Hermite polynomials are related to the problem of an evaluating square root of Casimir operators. So this phenomenon should be also studied on the algebraic level in order to clarify its physical nature.

Finally, as was suggested by the referee, it will be of interest to employ the Darboux transformation to equations (14) and (20) to find out what kind of non-classical q-orthogonal polynomials are associated with them. In the limit when $q \to 1$ these q-polynomials should coincide with those, that appear upon applying the Darboux transformation to the stationary one-dimensional Schrödinger operator [18].

Acknowledgments

We are grateful to N M Atakishiyev for suggesting to us the problem and a helpful discussion.

References

[1] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581–8
[2] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873–8
[3] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415–9
[4] Gasper G and Rahman M 1990 *Basic Hypergeometric Functions* (Cambridge: Cambridge University Press)
[5] Andrews G E, Askey R and Roy R 1999 *Special Functions* (Cambridge: Cambridge University Press)
[6] Nikiforov A F, Suslov S K and Uvarov V B 1991 *Classical Orthogonal Polynomials of a Discrete Variable* (Berlin: Springer)
[7] Landau L D and Lifshitz E M 1991 *Quantum Mechanics (Non-relativistic Theory)* (Oxford: Pergamon)
[8] Koekoek R and Swarttouw R F 1998 The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue *Report 98–17* Delft University of Technology, Delft (availabe from ftp.tudelft.nl)
[9] Koornwinder T H 2006 The structure relation for Askey–Wilson polynomials *Preprint* math.CA/0601303 (*J. Comput. Appl. Math.* at press)
[10] Atakishiyev N M and Suslov S K 1991 *Theor. Math. Phys.* **85** 1055–62
[11] Ismail M E H 2005 *Classical and Quantum Orthogonal Polynomials in One Variable* (Cambridge: Cambridge University Press)
[12] Akhiezer N I and Glazman I M 1961 *The Theory of Linear Operators in Hilbert Spaces* (New York: Ungar)

[13] Atakishiyeva M K, Atakishiyev N M and Villegas-Blas C 1998 *J. Comput. Appl. Math.* **99** 27–35
[14] Atakishiyeva M K and Atakishiyev N M 2001 *Phys. At. Nuclei* **64** 2086–92
[15] Askey R 1989 *q-Series and Partitions (The IMA Volumes in Mathematics and Its Applications* vol 18*)* ed D Stanton (New York: Springer) pp 151–8
[16] Ismail M E H and Masson D R 1994 *Trans. Am. Math. Soc.* **346** 63–116
[17] Atakishiyev N M and Klimyk A U 2006 *Mod. Phys. Lett.* A **29** 2205–16
[18] Samsonov B F and Ovcharov I N 1995 *Izvestiya Vysshix Uchebnix Zavedenii, Fizika* **4** 58–65 (in Russian)